A. Project Summary

Although the physical world is quantum mechanical in nature, our perceptions of it are rooted in classical mechanics. Thus one is often confronted with the problem of constructing a quantum formulation of a system from a knowledge of a classical approximation to it. This process is called "quantization," and over the years many different quantization schemes have been developed.

Unfortunately, quantization is not a straightforward proposition. The main difficulty stems from the discovery, over fifty years ago, by Groenewold and Van Hove of an "obstruction" to quantization. Their "no-go theorem" asserts that *in principle* it is impossible to consistently quantize every classical observable on the phase space of a freely moving particle in a physically meaningful way, regardless of which quantization procedure is employed. Over the past few years, the principal investigator and his collaborators proved that similar results hold for a wide variety of phase spaces. But no-go theorems are not universally valid; the principal investigator has recently found several phase spaces which admit consistent quantizations.

The main goals of this proposal are:

(*i*) To delineate the circumstances under which such obstructions will appear, and to understand the mechanisms which produce them. Already substantial progress has been made: the principal investigator and his collaborators have proven, in quite general circumstances, the existence of obstructions to obtaining both finite- and infinite-dimensional quantizations of compact phase spaces as well as finite-dimensional quantizations of noncompact phase spaces.

(*ii*) To compute, when an obstruction does exist, the largest subalgebras of observables that can be consistently quantized along with all their possible quantizations. While this can sometimes be done in specific examples, little is known in general. Moreover, this line of investigation has brought to light previously unknown quantizations of various physical systems, and it is important to determine their physical significance.

(*iii*) To refine extant quantization procedures, or perhaps design new ones, which are adapted to the obstruction in the sense that they will automatically be able to quantize these subalgebras. Typically, standard techniques are able to quantize only relatively "small" subalgebras of observables.

(*iv*) The above results are valid for quantization procedures which are "Hilbert space-based." It is therefore important to determine to what extent they remain valid in other approaches, such as deformation quantization, which employs a somewhat "looser" notion of quantization. Anecdotal evidence suggests that the obstructions encountered in the Hilbert space-based context do indeed carry over to deformation quantization as well. A key aim of the proposed research is to elucidate this connection.

The proposed research is in an area of active interest to mathematicians and physicists alike. From a mathematical standpoint, this research will lead to interesting structural insights into the Poisson algebras of classical systems and their representations. Physically, this research will substantially aid in clarifying the correspondence between classical and quantum mechanics in general, and in particular will enhance our understanding of quantizations of specific classical systems.

C. Project Description

I. Results from Prior NSF Research

a.	**NSF Award Number:**	DMS 96-23083
	Amount:	\$73,493
	Period of Support:	6/1/96–5/31/00
b.	**Title of Project:**	Studies in Quantization Theory

c. Summary of Results of Completed Work:

My previous NSF-funded research consisted mainly of a study of obstructions in quantization theory. I also continued earlier work on analyzing the mathematical structure of classical field theory, and initiated a project applying geometric quantization to quantum chemistry. I briefly summarize the results from each component in turn. References cited below are to the list in Section D.

(*i*) *OBSTRUCTIONS IN QUANTIZATION THEORY*

In 1946 H. Groenewold and in 1951 L. van Hove proved theorems to the effect that it is impossible to quantize the Poisson algebra of polynomials on $\mathbf{R}^{2n}$ in such a way that the Heisenberg subalgebra $\mathrm{h}(2n)$ consisting of linear polynomials is irreducibly represented [Gro, VH]. This result has led people to conjecture, roughly speaking, that *it is impossible to consistently quantize the entire Poisson algebra of a symplectic manifold subject to an irreducibility requirement.* I refer the reader to Section C.II following for the background of this problem, as well as a more complete discussion of the issues involved here.

My research has been concerned with understanding the origins of Groenewold-Van Hove type obstructions as well as delineating the circumstances under which they will appear. The first goal is to correctly set up the problem, which involves defining what is meant by a "basic algebra of observables" (analogous to $\mathrm{h}(2n)$ on $\mathbf{R}^{2n}$) which is to be irreducibly represented, as well as giving a precise meaning to the notion of "quantization." All this has been satisfactorily accomplished [Go5]. The precise definitions are somewhat technical, and so are deferred until §C.II; here I shall use these terms intuitively. So let $\mathfrak{b}$ be a basic algebra of observables on a symplectic manifold M, and let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$ containing $\mathfrak{b}$. A *quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ is a Lie representation of $\mathcal{O}$ on a Hilbert space which (amongst other things) is irreducible when restricted to $\mathfrak{b}$. The *Groenewold-Van Hove problem* is to determine whether $\mathcal{O}$ can be consistently quantized and, if not, to find the largest Lie subalgebras of $\mathcal{O}$ that can be quantized and to explicitly construct all their possible quantizations.

While the prevailing expectation is that "no-go" results should hold in a wide range of situations, there has been little work done in this direction since Groenewold and Van Hove (possibly because the mathematical tools with which to tackle this problem weren't available). Thus the first step is to check whether the conjecture above is in fact true in specific examples.

To this end, H. Grundling, C. Hurst, and I considered the quantization of S^2, thought of as the 'internal' phase space of a nonrelativistic particle with spin [GGH]. We chose this example since it is structurally quite different from $\mathbf{R}^{2n}$. We found that a no-go theorem indeed holds for S^2. Let P_k (resp. P^k) denote the space of all spherical harmonics of degree k (resp. of degree at most k), and let P be the Poisson algebra of all spherical harmonics on S^2. Note that P_1 is isomorphic to su(2), which is the basic algebra in this instance, and P^1 to u(2).

Theorem 1 *There is no nontrivial quantization of (P, P_1). Furthermore, no nontrivial quantization of P_1 can be extended beyond P^1 in P.*

Thus there is an obstruction for S^2 which is similar to that for $\mathbf{R}^{2n}$, except it is more 'severe.' Since all possible quantizations of P^1 are explicitly known—these being the familiar spin representations of u(2)—we have thus completely solved the Groenewold-Van Hove problem for S^2.

H. Grundling and I then turned our attention to T^*S^1, taking as a basic algebra the Euclidean algebra e(2) $= \mathrm{span}\{\ell, \sin\theta, \cos\theta\}$, where ℓ is the angular momentum conjugate to θ. We showed that it too exhibits an obstruction, and completely solved the Groenewold-Van Hove problem in this case as well [GGru1].

Between $\mathbf{R}^{2n}$, S^2, and T^*S^1, we now have examples which exhibit a wide range of behaviors: from compact to contractible, and whose quantizations are finite- as well as infinite-dimensional. Thus it would seem reasonable to conjecture:

> *Let $\mathfrak{b}$ be a basic algebra of observables on the symplectic manifold M, and let $P(\mathfrak{b})$ be the Poisson algebra of polynomials generated by $\mathfrak{b}$. Then there is no nontrivial quantization of the pair $(P(\mathfrak{b}), \mathfrak{b})$.*

Surprisingly, this conjecture turns out to be *false*. Consider the torus, thought of as $\mathbf{R}^2/\mathbf{Z}^2$. The natural choice for a basic algebra in this instance is the Lie algebra $\mathfrak{t}$ generated by $\{\sin 2\pi x, \cos 2\pi x, \sin 2\pi y, \cos 2\pi y\}$. (Thus $\mathfrak{t}$ consists of trigonometric polynomials of mean zero.) Let $\mathcal{H}$ be the Hilbert space of quasi-periodic functions

$$\phi(x+m, y+n) = e^{2\pi i m y}\phi(x, y), \qquad m, n \in \mathbf{Z}$$

which are square-integrable over $(0, 1]^2$. For each $f \in C^\infty(T^2)$, define a (self-adjoint) operator $\mathcal{Q}(f)$ on $\mathcal{H}$ by

$$\mathcal{Q}(f) = -i\hbar\left(\frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y} - \frac{i}{\hbar}x\right) - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right) + f. \tag{1}$$

Then I have proven the following "go" theorem [Go4]:

Theorem 2 *$\mathcal{Q}$ is a quantization of $(C^\infty(T^2), \mathfrak{t})$ on $\mathcal{H}$.*

Thus there is no Groenewold-Van Hove obstruction to quantizing the torus. However, this example is crucially different than the previous ones: $\mathbf{R}^{2n}$, S^2, and T^*S^1 all carry finite-dimensional basic algebras, but T^2 does not [GGG]. Consequently the irreducibility requirement on $\mathfrak{t}$ is comparatively much weaker than in the other examples, and so this is likely the reason why $C^\infty(T^2)$ can be consistently quantized. So perhaps the conjecture above will hold if the basic algebra is required to be finite-dimensional?

Surprisingly, the answer is "No!" A counterexample is provided by $T^*\mathbf{R}_+ = \{(q, p) \in \mathbf{R}^2 \mid q > 0\}$ with the "affine" basic algebra $\mathrm{a}(1) = \mathrm{span}\{pq, q^2\}$. Several inequivalent quantizations of the polynomial algebra $P(\mathrm{a}(1))$ on $L^2(\mathbf{R}_+, dq/q)$ are explicitly constructed in [GGra1], cf. §C.II. Another simple example is either of the nilpotent coadjoint orbits $N_\pm$ in $\mathrm{sl}(2)^*$ with basic algebra sl(2) [GGra2].

At this point we have exhaustively studied several examples, with varying outcomes as to the existence of obstructions to quantization. I summarize these examples below.

M	*Type*	$\mathfrak{b}$	*Type*	*Representations*	*Obstruction?*
$\mathbf{R}^{2n}$	noncompact	h$(2n)$	nilpotent	infinite-dimensional	Yes
S^2	compact	su(2)	simple	finite-dimensional	Yes
T^2	compact	$\mathfrak{t}$	infinite-dimensional	infinite-dimensional	No
T^*S^1	noncompact	e(2)	solvable	infinite-dimensional	Yes
$T^*\mathbf{R}_+$	noncompact	a(1)	solvable	infinite-dimensional	No
$N_\pm$	noncompact	sl(2)	simple	infinite-dimensional	No

Table 1: Summary of known examples.

There is no obvious pattern. Moreover, our treatment of these examples relied heavily on a detailed knowledge of the representations of the relevant basic algebras, and involved complex (and often computer-aided) calculations. To obtain general results on the occurrence of obstructions, it is clear that we need to suppress the representational aspects, and focus instead on the Lie and Poisson structures of basic algebras and the polynomial algebras they generate.

The first key results in this direction were due to Avez [Av1] and Ginzburg and Montgomery [GiM]. Inspired by their work, J. Grabowski, H. Grundling, and I were able to produce several no-go results for polynomial quantizations. Let $\mathfrak{b}$ be a *finite-dimensional* basic algebra of observables. We broke the analysis up into four cases, depending upon whether $\mathfrak{b}$ (or equivalently M) is compact and its representations are finite-dimensional. While space does not permit me to elaborate on the proofs of the following results, I remark that they rely to some extent on the work of Atkin [At] and Grabowski [Gra1, Gra2] on Poisson algebras. Furthermore, in §C.II I sketch a proof (of Theorem 9), which is fairly representative of the techniques involved here.

(*i*) *M Compact, Finite-dimensional Representations.* The main result is [GGG]:

Theorem 3 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a compact symplectic manifold M. There exists no nontrivial finite-dimensional Lie representation of $P(\mathfrak{b})$. In particular, there exists no nontrivial finite-dimensional quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Although not surprising on mathematical grounds, since $P(\mathfrak{b})$ is "large," this theorem does have physical import, as one expects the quantization of a compact phase space to yield a *finite*-dimensional Hilbert space.

(*ii*) *M Compact, Infinite-dimensional Representations.* We reduce this to the previous case by observing that a finite-dimensional basic algebra on a compact symplectic manifold must itself be compact, and hence its irreducible representations must all be finite-dimensional. Then Theorem 3 applies, and we have [GGG]

Corollary 4 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a compact symplectic manifold M. There exists no nontrivial quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Thus, there is an obstruction to polynomially quantizing a compact symplectic manifold *regardless* of the dimensionality of the representation. I emphasize, however, that the finite-dimensionality of $\mathfrak{b}$ is crucial; as the torus shows, Corollary 4 fails if this assumption is removed.

(*iii*) *M Noncompact, Finite-dimensional Representations.* On physical grounds one expects a quantization of a noncompact M, if it exists, to be infinite-dimensional. This is what we have rigorously proved in [GGru2]. The crucial observation is:

Theorem 5 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a noncompact symplectic manifold. Then $\mathfrak{b}$ has no faithful finite-dimensional representations by Hermitian matrices.*

Since by definition every quantization of $(\mathcal{O}, \mathfrak{b})$ must be faithful on $\mathfrak{b}$, we conclude that *there is no nontrivial finite-dimensional quantization of $(\mathcal{O}, \mathfrak{b})$ on a noncompact symplectic manifold*, where $\mathcal{O}$ is *any* Lie algebra containing $\mathfrak{b}$. Combining this with Theorem 3 we can now assert—roughly speaking—that no symplectic manifold with a (finite-dimensional) basic algebra has a finite-dimensional quantization.

(*iv*) *M Noncompact, Infinite-dimensional Representations.* Since it is difficult to treat this case inclusively, we have begun by breaking it into subcases depending on the type of basic algebra. The simplest subcase to consider is when $\mathfrak{b}$ is nilpotent. Then we have shown, building on the results of Wildberger [Wi] and others, that M must be symplectomorphic to some $\mathbf{R}^{2n}$. (Note, though, that $\mathfrak{b}$ need not be isomorphic to $\mathrm{h}(2n)$.) Furthermore, we have established the following generalization of the classical Groenewold-Van Hove theorem [GGra1].

Theorem 6 *Let $\mathfrak{b}$ be a finite-dimensional nilpotent basic algebra on a connected symplectic manifold M. There exists no quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

So far we have encountered obstructions in every instance. The present case is the exception: As mentioned previously, we have shown that there exists a polynomial quantization of $T^*\mathbf{R}_+$ with the affine basic algebra a(1). (Note that a(1) is the simplest solvable Lie algebra which is not nilpotent!) However, the behavior exhibited by this example is not characteristic of solvable algebras, since the Euclidean algebra e(2) is also solvable yet the quantization of T^*S^1 is obstructed.

At the other extreme, we have also begun studying semisimple basic algebras. Here again, we encounter examples which admit polynomial quantizations [GGra2]:

Theorem 7 *Let M be a nilpotent adjoint orbit in the finite-dimensional semisimple Lie algebra $\mathfrak{b}$. If M admits $\mathfrak{b}$ as a basic algebra, then there exists a nontrivial quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

On the other hand, we expect that the quantizations of non-nilpotent orbits will be obstructed although we do not yet have a proof of this.

Thus we are able to obtain obstructions to quantizing $(P(\mathfrak{b}), \mathfrak{b})$ in three of these cases. And in the remaining case (viz. when $\mathfrak{b}$ is noncompact and the representation space is infinite-dimensional), there is no universal obstruction. In this gross sense, then, we have solved the problem of whether obstructions to quantization exist for polynomial quantizations.

Finally, to close the circle, it turns out that there is a technical gap in Groenewold's original proof [Gro]. This gap has been filled in [VH] (see also [AM]) by means of a certain functional analytic assumption. Although "small," this gap is nevertheless vexing, and its elimination in this manner is not entirely satisfactory. Recently [Go6], I have managed to rigorously prove Groenewold's theorem *without* introducing extra assumptions. In addition, when $n = 1$ I have listed the maximal quantizable Lie subalgebras of polynomial observables and classified their quantizations. Thus the Groenewold-Van Hove problem for polynomial quantizations on $\mathbf{R}^{2n}$ is completely solved for $n = 1$.

(ii) *GEOMETRIC QUANTIZATION AND QUANTUM CHEMSITRY*

I. Mladenov and I have been working on a project in quantum chemistry. It is an open problem to predict the rotational spectra of even simple molecules, except in certain highly symmetric situations. For instance, while in [LL] the low-lying energy eigenvalues of a polyatomic molecule can be computed using a recursive technique, there is no known closed form expression for these quantities. Using geometric quantization theory, however, we have managed to produce a such an expression for the energy eigenvalues in terms of elliptic integrals [GoMl]. We are currently working on simplifying this (complicated) result, and are investigating several special cases. In particular, for a diatomic molecule our formula reduces to

$$E = \frac{\hbar^2}{2I_1}\left(L + \frac{1}{2}\right)^2 + \frac{\hbar^2}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right)m^2,$$

where $I_1 = I_2 \neq I_3$ are the moments of inertia, L is the angular momentum quantum number, and m is the component of the angular momentum along the molecular axis.

Our formulæ are interesting in two regards. First, instead of the usual $\hbar^2 L(L+1)$ term, we have $\hbar^2(L+1/2)^2$. One can regard this either as a particular manifestation of the "metaplectic shift" in geometric quantization theory or as resulting from a "curvature correction" [Wo]. (In the context of angular momentum, this shift of $\hbar^2/4$ is referred to as the "Langer modification" [La].) Second, they imply that there should exist a rotational rest energy of $\hbar^2/8I_1$, much like the well-known vibrational zero-point energy. This shift may or may not be physically correct; in some instances, it gives the right answers (e.g. the vibrational rest energy), but in others it does not (e.g. spin). So we need to examine the rotational spectra of, say, diatomic molecules for evidence of such a rest energy.

We do not yet know if this is detectable. Determining this is likely to be difficult; in the realm of quantum chemistry, even the existence of the vibrational rest energy is a subtle effect which can be observed only via isotopic displacement [Her, §IV.2]. Complicating matters are other degrees of freedom (vibrational, internal) whose presence tend to "swamp" rotational effects in the energy spectra, as well as the fact that the phenomenological parameters describing molecules (moments of inertia, for instance) are imprecisely known.

(iii) *THE MATHEMATICAL STRUCTURE OF CLASSICAL FIELD THEORY*

The purpose of this research was to develop and exploit connections between initial value constraints and gauge transformations in classical field theories. To a substantial degree I, along with collaborators J. Isenberg and J. Marsden, have succeeded in this. We have shown that many different and apparently unrelated facets of field theory can be tied together and understood in a fundamentally new way using multisymplectic techniques.

Our research program consists of four components: (A) a covariant analysis of field theories; (B) a space + time decomposition of the covariant formalism followed by an initial value analysis of field theories; (C) a study of the relations between gauge symmetries and initial value constraints; and (D) the derivation of the so-called "adjoint formalism," which constitutes the starting point for investigations into the structure of the space of solutions of the Euler-Lagrange equations, linearization stability, quantization, and related questions.

This project is virtually finished. Part A has already appeared as a preprint [GIM1], and Part B will appear this winter [GIM2]. Part C is well understood in the case of purely first class theories, but there are a number of issues which require further study when second class constraints are present. Finally, the research for Part D has also been completed. There have been several significant spin-offs from this work as well, including [Go2, Go3, GoMa, MPS, MS]. When assembled, the results will appear as a research monograph of about 250 pages, which has already been accepted for publication in the M.S.R.I. series (Cambridge University Press).

Human Resources Statement

Interest in the research pursued in this grant has motivated a former graduate student in mathematics, Jason Hanson, to collaborate with me in on topics in classical field theory. In particular, he has been working out the details of the adjoint formalism for second order Einstein gravity. A graduate student, Bryon Kaneshige, has begun working with me on the problem of quantizing semisimple basic algebras, and plans to do his dissertation in this general area.

d. Publications Resulting from NSF Award:

1. Gotay, M.J., Grabowski, J., & Grundling, H.B. [2000] An obstruction to quantizing compact symplectic manifolds. *Proc. Amer. Math. Soc.* **128**, 237–243.
2. Gotay, M.J. & Grabowski, J. [1999] On quantizing nilpotent and solvable basic algebras. Submitted. Preprint math-ph/9902012.
3. Gotay, M.J. [1999] On the Groenewold-Van Hove problem for $\mathbf{R}^{2n}$. *J. Math. Phys.* **40**, 2107-2116.
4. Gotay, M.J. [1999] Obstructions to quantization. To appear in: *The Juan Simo Memorial Volume*, Marsden, J.E. & Wiggens, S., Eds. (Springer, New York). Preprint math-ph/9809011.
5. Gotay, M.J. & Grundling, H. [1999] Nonexistence of finite-dimensional quantizations of a noncompact symplectic manifold. In: *Differential Geometry and Applications*, Kolář, I. et al., Eds. (Masaryk University, Brno), 593–596.
6. Gotay, M.J. [1998] *Symplectic Geometry*. Editor. *Diff. Geom. Appl.* **9**(1-2).
7. Gotay, M.J., Isenberg, J.A., & Marsden, J.E. [1997] Momentum maps and classical relativistic fields, I: Covariant field theory. Submitted. physics/9801019.
8. Gotay, M.J. & Grundling, H.B. [1997] On quantizing T^*S^1. *Rep. Math. Phys.* **40**, 107–123.
9. Gotay, M.J. & Demaret, J. [1997] Some remarks on singularities in quantum cosmology. *Nuc. Phys. B* (Proc. Suppl.) **57**, 227–230.
10. Gotay, M.J., Grundling, H., & Tuynman, G.T. [1996] Obstruction results in quantization theory. *J. Nonlinear Sci.* **6**, 469–498.
11. Gotay, M.J., Grundling, H., & Hurst, C.A. [1996] A Groenewold-Van Hove theorem for S^2. *Trans. Amer. Math. Soc.* **348**, 1579–1597.
12. Gotay, M.J. [1995] On a full quantization of the torus. In: *Quantization, Coherent States and Complex Structures*, Antoine, J.-P. et al., Eds. (Plenum, New York) 55–62.

f. Relation of Completed Work to Proposed Work:

My proposed research program is a straightforward continuation of my current NSF supported research.

II. Proposed Research Project: *Obstructions in Quantization Theory*

My proposed research project is a direct followup to my current NSF supported research on the existence of obstructions to quantization (cf. Section C.I). The main goals of this proposal are:

• To delineate the circumstances under which such obstructions will appear, and to understand the mechanisms which produce them. Already substantial progress has been made: My collaborators and I have proven, in quite general circumstances, the existence of obstructions to obtaining both finite- and infinite-dimensional quantizations of compact phase spaces as well as finite-dimensional quantizations of noncompact phase spaces. From a practical standpoint, determining the extent to which a quantization is internally consistent is an essential part of the quantization program; when one pushes quantization too far all sorts of problems are known to arise. Thus one needs to know just how far "too far" is; specifically, one needs

• To compute, when an obstruction does exist, the largest subalgebras of observables that can be consistently quantized. While this can sometimes be done in specific examples, little is known in general. Once this is accomplished, the next step is to classify all their possible quantizations. This line of investigation has brought to light previously unknown quantizations of several classical systems, and it is important to determine their physical significance. Again, while this can often be carried out by hand in particular cases, it would be preferable

• To refine extant quantization procedures, or perhaps design new ones, which are adapted to the obstruction in the sense that they will automatically be able to quantize these largest subalgebras. Typically, standard techniques are able to quantize only relatively "small" subalgebras of observables, and run into difficulties when extended beyond these classes of observables. Such a quantization procedure would be of obvious value when confronted with complicated and poorly understood classical models which need to be quantized.

• My emphasis thus far has been on quantization procedures which are "Hilbert space-based." It is therefore important to determine to what extent they remain valid in other approaches, such as deformation quantization, which employs a somewhat "looser" notion of quantization. Anecdotal evidence suggests that the obstructions encountered in the Hilbert space-based context do indeed carry over to (strict) deformation quantization as well. A key aim is to elucidate this connection.

The proposed research is in an area of active interest to mathematicians and physicists alike. From a mathematical standpoint, this research will lead to interesting structural insights into the Poisson algebras of classical systems and their representations. Physically, it will substantially aid in clarifying the correspondence between classical and quantum mechanics in general, and in particular will enhance our understanding of quantizations of specific classical systems.

I will discuss each point in turn after giving the background of the problem.

Quantization has always been one of the great mysteries of mathematical physics, dating back to the beginnings of this century. There are at this point many ways of quantizing a classical mechanical system, including the physicists' original "canonical quantization" [Di] (and its modern mathematical formulations, such as geometric quantization theory [Ki, Wo]), Weyl quantization [Fo], path integral quantization [GJ] and the more recent deformation quantization [BFFLS, Ri2].

Regardless of which quantization procedure one favors, it is generally accepted—although not necessarily well substantiated—that quantization is an ill-defined procedure, which is inherently incapable of consistently quantizing all classical systems. While there is certainly no extant quantization procedure which works well in all circumstances, and while there some evidence supporting this assertion, it nonetheless bears closer scrutiny.

That any specific quantization scheme has shortcomings probably reflects the fact that there is a myriad of quantum theories, all of which have the same classical limit. How then is a quantization procedure to select the physically correct quantum theory amongst these possibilities?

But there are deeper, underlying problems, which are for the most part independent of the particular method of quantization employed. In this context the conventional wisdom is that it is impossible to "fully" quantize a given classical system, in a way which is consistent with the physicists' Schrödinger quantization of $\mathbf{R}^{2n}$. (I will make this somewhat nebulous statement precise below.) In other words, the assertion is that there exists a universal "obstruction" which implies that one must settle for something less than a complete and consistent quantization of *any* system. Each quantization procedure listed above exhibits this defect in specific examples.

That there are problems in quantizing even simple systems was observed very early on. One difficulty was to identify the analogue of the multiplicative structure of the classical observables in the quantum formalism. For instance, consider the quantization of the phase space $\mathbf{R}^{2n}$ with canonical coordinates $\{q^i, p_i, i = 1, \dots, n\}$. For simple observables the the "product $\to$ anti-commutator" rule works well. But for more complicated observables (say, ones which are quartic polynomials in the positions and momenta), this rule itself becomes ambiguous and inconsistencies arise (see [Fo, §1.1] for a discussion of these factor-ordering ambiguities). Of course this, in and by itself, might only indicate the necessity of coming up with some subtler symmetrization rule. But attempts to construct a quantization map also conflicted with Dirac's "Poisson bracket $\to$ commutator" rule. This was implicitly acknowledged by Dirac [Di, p. 87], where he made the now famous hedge:

> "*The strong analogy between the quantum P.B. . . . and the classical P.B. . . . leads us to make the assumption that the quantum P.B.s, or at any rate the simpler ones of them, have the same values as the corresponding classical P.B.s.*"

In any case, as a practical matter, one was forced to limit the quantization to relatively 'small' subalgebras of observables which could be handled without ambiguity (e.g., polynomials which are at most quadratic in the p_i and the q^i, or observables which are at most affine functions of the momenta).

Then, in 1946, Groenewold [Gro] showed that the search for an "acceptable" quantization map was futile. His "no-go" theorem states that one cannot consistently quantize all polynomials in the q^i and p_i on $\mathbf{R}^{2n}$, subject to the requirement that the q^i and p_i be irreducibly represented. Subsequently Van Hove [VH] refined Groenewold's result. (For discussions of these and related results, see [AM, Ch1, Fo, Go1, Go6, GS, Jo] and references contained therein.) Thus it is *in principle* impossible to quantize – by *any* means – every polynomial observable on $\mathbf{R}^{2n}$ in a way consistent with Schrödinger quantization (which, according to the Stone-Von Neumann theorem, is the import of the irreducibility requirement on the p_i and q^i). At most, one can consistently quantize certain subalgebras of observables such as those mentioned in the preceding paragraph.

Examination of specific quantization procedures provides corroboration for Groenewold's result. Consider for instance geometric quantization theory, in which context the only observables which are quantizable *a priori* are those whose Hamiltonian vector fields preserve a given polarization [Bl1, Wo]. While this does not preclude the possibility of quantizing more general observables, attempts to quantize observables outside this class usually result in inconsistencies. In *all* instances, the set of *a priori* quantizable observables forms a proper Lie subalgebra of the Poisson algebra of the given symplectic manifold. This observation leads one to expect that Groenewold-Van Hove obstructions to quantization should be ubiquitous.

The principal question I wish to investigate is whether this is actually true: Under what conditions do no-go theorems hold for general symplectic manifolds? For nearly half a century Groenewold's theorem was essentially the only result along these lines (possibly because the mathematical tools with which to tackle this problem weren't available). Then, during the past few years, H. Grundling, C. Hurst, and I proved that there are obstructions to quantizing both the sphere S^2 [GGH] and the cylinder T^*S^1 [GGru1]. At the same time, I showed that no-go theorems are *not* universal: There are no obstructions to quantizing either the torus T^2 [Go4] or $T^*\mathbf{R}_+$ [GGra1].[1] An important point, therefore, is to understand the mechanisms which are responsible for these divergent outcomes.

With several examples now in hand, it is possible to probe deeper and try to understand the underlying reasons for the existence (or nonexistence) of obstructions to quantization. To set the stage for this, I introduce some terminology. Let M be a connected symplectic manifold, with Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is the Poisson bracket.

Definition 1 Let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$. A *prequantization* of $\mathcal{O}$ is a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space Op(D) of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space $\mathcal{H}$, such that for all $f, g \in \mathcal{O}$,

[1] These results are described in "Results from Prior NSF Research," §C.I.

(Q1) $\mathcal{Q}(\{f, g\}) = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$,

(Q2) if $\mathcal{O}$ contains the constant function 1, then $\mathcal{Q}(1) = I$, and

(Q3) if the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D.

If $\mathcal{O} = C^\infty(M)$, the prequantization is said to be *full*. $\mathcal{Q}$ is *nontrivial* provided codim ker $\mathcal{Q} > 1$.

Prequantizations in this broad sense are usually easy to construct [Ch3, Ur]. Van Hove was the first to prequantize $C^\infty(\mathbf{R}^{2n})$ [VH]. Notice that no assumptions are made regarding the multiplicative structure on $\mathcal{O}$ vis-à-vis $\mathcal{Q}$.

Unfortunately, prequantization representations of the entire Poisson algebra of a symplectic manifold tend to be flawed physically. For instance, the Van Hove prequantization of $\mathbf{R}^{2n}$ is not unitarily equivalent to the Schrödinger representation of the Heisenberg group H($2n$) [Bl1], while the Kostant-Souriau prequantizations of S^2 do not reproduce the standard spin representations of SU(2) [Wo]. In both examples the prequantization Hilbert spaces are 'too big.' The main problem is how to remedy this, in other words, how to modify the notion of a prequantization so as to yield a genuine *quantization*.

Several notions of what constitutes a quantization map can be found in the literature. Some versions (e.g, [Ki, Is]) take a certain "basic algebra" $\mathfrak{b} \subset C^\infty(M)$ of observables and then define a quantization as a prequantization which is irreducible on $\mathfrak{b}$. For example when $M = \mathbf{R}^{2n}$, $\mathfrak{b}$ is the Heisenberg algebra h($2n$), and when $M = S^2$, one would take $\mathfrak{b} = \mathrm{su}(2)$.

A different definition of a quantization is a prequantization $\mathcal{Q}$ which satisfies a "Von Neumann rule," that is, some given relation between the multiplicative structure of $C^\infty(M)$ and operator multiplication on $\mathcal{H}$. There are many different types of such rules [KLZ, KS, VN], the simplest being of the form:

$$\mathcal{Q}(\varphi \circ f) = \varphi(\mathcal{Q}(f))$$

for some distinguished observables $f \in C^\infty(M)$ and smooth functions $\varphi \in C^\infty(\mathbf{R})$.

A third type of quantization is obtained by polarizing a prequantization representation [Bl1, Wo]. In the context of Poisson algebras, a 'polarization' is a maximal commuting Lie subalgebra $\mathcal{A}$ (of the complexification) of $C^\infty(M)$. Then one requires for the quantization map $\mathcal{Q}$ that the image $\mathcal{Q}(\mathcal{A})$ be maximally commuting as operators in an appropriate technical sense.

These three approaches to a quantization map are not independent; in fact, there exist subtle connections between them which are not well understood. But at the core of each approach is the imposition—in some form—of an irreducibility requirement which is used to 'cut down' the prequantization representation. Since this is most apparent in the first approach, I will concentrate henceforth on it.

The first order of business is to determine exactly what constitutes a "basic algebra" of observables. After much trial and error, the following definition seems to be the most appropriate.

Definition 2 A *basic algebra of observables* $\mathfrak{b}$ is a Lie subalgebra of $C^\infty(M)$ such that:

(B1) $\mathfrak{b}$ is finitely generated,

(B2) the Hamiltonian vector fields X_f, $f \in \mathfrak{b}$, are complete,

(B3) $\mathfrak{b}$ is transitive and separating, and

(B4) $\mathfrak{b}$ is a minimal Lie algebra satisfying these requirements.

I briefly elaborate on this definition; a more detailed discussion can be found in [Go5]. First, it is natural to require that a basic algebra be finite-dimensional, but this turns out to be overly restrictive. For example, in [GGG] it is shown that the torus has no finite-dimensional basic algebras. The obvious choice for a basic algebra on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ is the Lie algebra $\mathfrak{t}$ of trigonometric polynomials of mean zero [Go4], which is of course infinite-dimensional. However, note that $\mathfrak{t}$ is generated by the finite set

$$\{\sin 2\pi x,\ \cos 2\pi x,\ \sin 2\pi y,\ \cos 2\pi y\},$$

so that one still has a finite number of “basic observables” with which to make measurements. I therefore do not demand that $\mathfrak{b}$ be finite-dimensional, and instead merely require (B1).

The completeness condition (B2) guarantees that a basic observable generates a one-parameter group of canonical transformations. In view of (Q3), it is the classical analogue of the requirement that an operator representing a physically observable quantity should be essentially self-adjoint, whence it generates a one-parameter group of unitary transformations.

Next consider the transitivity requirement in (B3), which means that $\{X_f(m) \mid f \in \mathfrak{b}\}$ spans $T_m M$ at every point. When $\mathfrak{b}$ is finite-dimensional, it together with (B2) enable one to integrate $\mathfrak{b}$ to a transitive action of the simply connected group B with Lie algebra $\mathfrak{b}$ on M, whence M is a Hamiltonian homogeneous space for B. The appeal of transitivity is that it is a classical version of an irreducible representation: Using the transitive action of B, one can obtain any classical state from any other one, in direct analogy with the fact that every nonzero vector in a Hilbert space $\mathcal{H}$ is cyclic for an irreducible unitary representation of B on $\mathcal{H}$. Furthermore, transitivity implies the following nondegeneracy condition, which is reminiscent of the quantum mechanical notion of “operator irreducibility”: $\{f, g\} = 0$ for all $f \in \mathfrak{b}$ implies g is constant.

As part of (B3) I also require that $\mathfrak{b}$ globally separate classical states. This ensures that $\mathfrak{b}$ accurately reflects the topology of M [Ve]. Without it, e.g., the Lie algebra $\mathfrak{t}$ defined above could equally well live on either $\mathbf{R}^2$ or T^2 (or even “halfway between,” on T^*S^1); measurements using $\mathfrak{t}$ could not distinguish amongst these phase spaces.

Finally, the minimality condition (B4) is crucial: The quantization of a pair $(\mathcal{O}, \mathfrak{b})$ with $\mathfrak{b}$ nonminimal in this sense can lead to physically incorrect results. For instance, without this condition it would often be possible to find full quantizations; indeed, it may happen that a prequantization representation is itself irreducible [VH, Ch3].

There is no guarantee that a given symplectic manifold will carry a basic algebra. Indeed, the next proposition shows that those phase spaces which admit basic algebras form a quite restricted class [Go5].

Proposition 8 *If a connected symplectic manifold M admits a finite-dimensional basic algebra $\mathfrak{b}$, then M is a coadjoint orbit in $\mathfrak{b}^*$.*

Despite this, M may still carry *infinite*-dimensional basic algebras, as happens for T^2. Not much is known regarding these, cf. [Is, §4.8.4] for further discussion.

With this definition of a basic set, I am now ready to state what I mean by “quantization” [Go5]. Let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$, and suppose that $\mathfrak{b} \subset \mathcal{O}$ is a basic algebra of observables. Two eminently reasonable requirements to place upon a quantization are irreducibility and integrability [BR, Fl, Is, Ki].

Definition 3 A *quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ is a prequantization $\mathcal{Q}$ of $\mathcal{O}$ on $\mathrm{Op}(D)$ satisfying

(Q4) $\mathcal{Q} \restriction \mathfrak{b}$ is irreducible,

(Q5) D contains a dense set of separately analytic vectors for a Lie generating set of $\mathcal{Q}(\mathfrak{b})$, and

(Q6) $\mathcal{Q} \restriction \mathfrak{b}$ is faithful.

Irreducibility is of course one of the pillars of the quantum theory, and we have already seen the necessity of requiring that a quantization map represent $\mathfrak{b}$ irreducibly. Regarding integrability, first consider the case when basic algebra is finite-dimensional. Then it is natural to demand that the Lie algebra representation $\mathcal{Q}(\mathfrak{b})$ be integrable to a unitary representation of B. That integrability will follow from the technical condition (Q5) is a theorem of Flato et al., cf. [Fl] and [BR, Ch. 11]. (There do exist nonintegrable representations, e.g. of the Heisenberg algebra [RS, p. 275]; however, none of them seem to have physical significance. (Q5) thus serves to eliminate these “spurious” representations.) If it happens that $\mathfrak{b}$ is infinite-dimensional, then there need not exist a Lie group having $\mathfrak{b}$ as its Lie algebra. Even if such a Lie group existed, integrability is far from automatic, and technical difficulties abound [ARS]. Thus I will not insist that a quantization be integrable in general. On the other hand, the analyticity requirement in (Q5) makes sense under all circumstances, and does guarantee integrability when $\mathfrak{b}$ is finite-dimensional, so I adopt it in lieu of integrability.

Finally, the faithfulness requirement (Q6) seems reasonable in that a classical observable can hardly be regarded as “basic” in a physical sense if it is in the kernel of a quantization map. In this case, it cannot be obtained in any classical limit from a quantum theory.

While considerable effort has gone into fine tuning these definitions, there are other requirements that one might wish to include. (A detailed discussion can be found in [Go5].) For example, one could strengthen the definition of a basic algebra by replacing (B3) by the condition that the Poisson algebra generated by $\mathfrak{b}$ be dense in $C^\infty(M)$. One could also modify the axioms for a quantization, for instance by adopting Souriau’s requirement that bounded classical observables should quantize to operators with bounded spectra [Zi]. Such changes could have profound consequences. In particular, if one modified (B3) as indicated, then both a(1) on $T^*\mathbf{R}_+$ and sl(2) on $N_\pm$ would be eliminated from the ranks of basic algebras. On the other hand, Proposition 8 shows that the given definition of basic algebra is already quite restrictive. I have framed the definitions so as to strike a compromise between these competing tendencies, while keeping physical considerations at the forefront. With regard to the latter, I emphasize that the definitions above are closely tailored to reflect what *physicists* mean when they speak of “basic observables” and “quantization.” In particular, there are many notions of quantization in the mathematical literature (in connection with quantum groups, spin^c structures and Riemann-Roch theorems, deformations of associative algebras, etc.) which, however, are often only vaguely related to “quantization” in the physical sense. Thus, from a mathematical standpoint, I take “quantization” to mean basically the process of obtaining certain types of representations of (Lie subalgebras of) Poisson algebras.

I am for the most part interested in the existence of polynomial quantizations, i.e., quantizations of $(P(\mathfrak{b}),\mathfrak{b})$ where $P(\mathfrak{b})$ is the Poisson algebra of polynomials generated by $\mathfrak{b}$. Let $P^k(\mathfrak{b})$ denote the subspace of polynomials of minimal degree at most k. (Since $P(\mathfrak{b})$ is not necessarily freely generated by $\mathfrak{b}$ as an associative algebra, the notion of “degree” may not be well-defined, but that of “minimal degree” is.) In the cases when degree does make sense, let $P_k(\mathfrak{b})$ denote the subspace of homogeneous polynomials of degree k. I then also introduce $P_{(k)}(\mathfrak{b}) = +_{l\geq k}P_l(\mathfrak{b})$. When $\mathfrak{b}$ is fixed in context, I simply write $P = P(\mathfrak{b})$, etc.

Table 2 following provides a summary of what is known about polynomial quantizations in general (cf. §C.I). For the time being, assume that $\mathfrak{b}$ is finite-dimensional.

	Finite-dimensional quantizations	*Infinite-dimensional quantizations*
M compact	No	No
M noncompact	No	Sometimes, but not always

Table 2: Do quantizations of $(P(\mathfrak{b}),\mathfrak{b})$ exist?

We are thus left with trying to understand the noncompact, infinite-dimensional case, which is naturally the most interesting and difficult one. Here one has little control over either the types of basic algebras that can appear (in examples they range from nilpotent to simple), the structure of the polynomial algebras they generate (the codimension of the commutator ideal $\{P,P\}$ can be at least 0,1, or ∞), or their representations. It is instructive to compare this with what happens when M is compact: Then $\mathfrak{b}$ is compact semisimple and codim $\{P,P\} = 1$ [GGG]. Nonetheless, as discussed in §C.I, we are able to prove a no-go theorem when $\mathfrak{b}$ is nilpotent, and to construct counterexamples when it is solvable or semisimple.

Now observe that four of our examples fall into this category: $\mathbf{R}^{2n}$, T^*S^1, $T^*\mathbf{R}_+$, and $N_\pm$. The first two exhibit obstructions, while the last two do not. Comparing the behavior of these examples, as well as that of S^2, which is also obstructed, I attempt to extract the key features which govern the appearance of obstructions to a polynomial quantization. Of course, any conclusions that can be drawn at this point are necessarily tentative, due to the paucity of examples against which to test them. Nonetheless, some interesting observations can be made, which may prove helpful in subsequent investigations.

I think the key lies in an examination of how the polynomial quantizations of $T^*\mathbf{R}_+$ and $N_\pm$ come about, since they relies upon some structural properties of P which are not present in our other examples. Therefore I now discuss $T^*\mathbf{R}_+$ in detail ($N_\pm$ works similarly). Recall that the "affine" basic algebra on $T^*\mathbf{R}_+$ is

$$\mathrm{a}(1) = \mathrm{span}\{pq, q^2\}.$$

Upon writing $X = pq$, $Y = q^2$, the bracket relation becomes $\{X, Y\} = 2Y$. The polynomial algebra $P = \mathbf{R}[X, Y]$ is freely generated by $\mathfrak{b}$, and has the crucial feature that for each $k \geq 0$, the subspaces P_k are *ad*-invariant, i.e.,

$$\{P_1, P_k\} \subset P_k. \tag{2}$$

(Note that $P_1 = \mathrm{a}(1)$). Because of this $\{P_k, P_l\} \subset P_{k+l}$, whence each $P_{(k)}$ is a Lie ideal. We thus have the semidirect sum decomposition

$$P = P^1 \ltimes P_{(2)}. \tag{3}$$

Since $P_{(2)}$ is a Lie ideal, I can obtain a quantization $\mathcal{Q}$ of *all* of P simply by finding an appropriate representation of $P^1 = \mathbf{R} \oplus P_1$ and setting $\mathcal{Q}(P_{(2)}) = \{0\}$!

The simply connected covering group of a(1) is isomorphic to the group $\mathrm{A}_+(1) = \mathbf{R} \rtimes \mathbf{R}_+$ of orientation-preserving affine transformations of the line (hence the terminology). Since $\mathrm{A}_+(1)$ is a semidirect product its unitary representations can be generated by induction; I compute

$$\big(U_\pm(\nu, \lambda)\psi\big)(q) = e^{\pm i\nu q^2/\hbar}\psi(\lambda q)$$

on $L^2(\mathbf{R}_+, dq/q)$. These two representations (one for each choice of sign) are irreducible and inequivalent; moreover, they are the only irreducible infinite-dimensional unitary ones. Writing $\pi_\pm = -i\hbar\, dU_\pm$ I get the quantization(s) of a(1) on $L^2(\mathbf{R}_+, dq/q)$:

$$\pi_\pm(pq) = -i\hbar q\frac{d}{dq}, \quad \pi_\pm(q^2) = \pm q^2.$$

Extend these to P^1 by demanding that $\pi_\pm(1) = I$, and set $\mathcal{Q}_\pm = \pi_\pm \oplus 0$ (cf. (3)).[2] These are evidently prequantizations of P, by construction (Q4) and (Q5) are satisfied, and $\mathcal{Q}_\pm \restriction P_1 = \pi_\pm$ are clearly faithful. Thus $\mathcal{Q}_\pm$ are the required quantization(s) of (P, P_1).

What makes this example work? It is clear that this polynomial quantization exists because one cannot decrease degree *in* P by taking Poisson brackets. (That is (2) holds, as opposed to merely $\{P_1, P_k\} \subset P^k$.) On the other hand, a careful look at the derivations of the obstructions in [Go6, GGru1, GGH] for $\mathbf{R}^{2n}$, T^*S^1, and S^2, respectively, shows that the controlling factor is apparently that in these examples one *can* decrease degree in P by taking Poisson brackets.

There are two—and only two—circumstances under which taking Poisson brackets in $P(\mathfrak{b})$ can decrease degree:[3]

(D1) $1 \in \{P(\mathfrak{b}), P(\mathfrak{b})\}$, and

(D2) There exist *nonzero* Casimirs in the symmetric algebra $S(\mathfrak{b})$ over $\mathfrak{b}$.[4]

Condition (D2) implies that $P(\mathfrak{b})$ is not freely generated by $\mathfrak{b}$ as an associative algebra. Specifically, (D2) holds whenever $\mathfrak{b}$ is semisimple and has a nonzero compact ideal. At the other extreme, when $\mathfrak{b}$ is nilpotent, (D1) holds. Indeed, a nilpotent algebra has a center, and (B3) implies that this center consists of constants. An examination of the descending central series for $\mathfrak{b}$ then shows that $1 \in \{\mathfrak{b}, \mathfrak{b}\}$. In the examples, $\mathbf{R}^{2n}$ satisfies (D1) but not (D2),

[2] There are also quantizations for which $\mathcal{Q}_\pm(P_{(2)}) \neq \{0\}$, cf. [GGra1].

[3] *A priori*, a third circumstance would be if $1 \in \mathfrak{b}$. Using the minimality condition (B4), it is not difficult to prove that then $1 \in \{\mathfrak{b}, \mathfrak{b}\}$, so this is actually a subcase of (D1).

[4] A *Casimir* is an element of the Lie center of $S(\mathfrak{b})$ which has no constant term. If C is a Casimir, then by (B3) its projection to $P(\mathfrak{b})$ is a constant. Thus this condition means that this constant must be nonzero; in other words, when viewed as a function on $M \subset \mathfrak{b}^*$, C is nonvanishing.

S^2 satisfies (D2) but not (D1), and T^*S^1 satisfies both. On the other hand, both $T^*\mathbf{R}_+$ and $N_\pm$ satisfy neither condition.

On the basis of this "anecdotal" evidence, I propose that a general Groenewold-Van Hove theorem takes the form:

Conjecture 1 *Let M be a symplectic manifold with a finite-dimensional basic algebra $\mathfrak{b}$. Suppose that the polynomial algebra $P(\mathfrak{b})$ satisfies either* (D1) *or* (D2)*. Then there is no nontrivial quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Indeed, is possible to directly verify this conjecture under certain circumstances.

Theorem 9 *Conjecture* 1 *is valid when either M is compact or the representation space is finite-dimensional.*

I sketch the proof so as to give a flavor of the techniques involved, which are fairly representative. Restrict attention to the case of finite-dimensional representations; the general case can easily be reduced to this. So suppose $\mathcal{Q}$ is a quantization of $P(\mathfrak{b})$ on a finite-dimensional Hilbert space, whence $\ker \mathcal{Q}$ has finite codimension in $P(\mathfrak{b})$.

As $\mathcal{Q}(\mathfrak{b})$ consists of hermitian matrices, $\mathcal{Q}$ is completely reducible. Since by (Q6) $\mathcal{Q}$ is also faithful, $\mathfrak{b}$ is reductive. As a consequence, $\mathfrak{b}$ splits as the Lie algebra direct sum $\mathfrak{z} \oplus \mathfrak{s}$, where $\mathfrak{z}$ is the center of $\mathfrak{b}$ and $\mathfrak{s}$ is semisimple. Now transitivity implies that any function which Poisson commutes with every element of $\mathfrak{b}$ must be a constant, so that $\mathfrak{z} \subseteq \mathbf{R}$. But if equality holds then $\mathfrak{s}$ would be a separating transitive subalgebra, thereby violating (B4). Thus $\mathfrak{z} = \{0\}$ and $\mathfrak{b}$ is semisimple.

I need the following result, which is proven in [GGG].

Lemma 1 *If L is a finite-codimensional Lie ideal of an infinite-dimensional Poisson algebra $\mathcal{P}$ with identity, then either L contains the commutator ideal $\{\mathcal{P}, \mathcal{P}\}$ or there is a maximal finite-codimensional associative ideal J of $\mathcal{P}$ such that $\{\mathcal{P}, \mathcal{P}\} \subseteq J$.*

Now apply this Lemma to $L = \ker \mathcal{Q} \subset P(\mathfrak{b})$. First suppose that $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq L$. Then semisimplicity gives $\mathfrak{b} = \{\mathfrak{b}, \mathfrak{b}\} \subset L$, and so $\mathcal{Q} \restriction \mathfrak{b} = 0$, which contradicts (Q6).

Thus there must exist a maximal finite-codimensional associative ideal J in $P(\mathfrak{b})$ with $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq J$. If ρ is the projection $S(\mathfrak{b}) \to P(\mathfrak{b})$, then $K = \rho^{-1}(J)$ is a maximal finite-codimensional associative ideal in $S(\mathfrak{b})$ with $\{S(\mathfrak{b}), S(\mathfrak{b})\} \subseteq K$. Since $\mathfrak{b} = \{\mathfrak{b}, \mathfrak{b}\} \subset \{S(\mathfrak{b}), S(\mathfrak{b})\} \subseteq K$, and since $1 \notin K$ (as J, and thus K is proper), it follows that K is the associative ideal generated by $\mathfrak{b}$.

If (D1) holds then $1 \in J$, which cannot be. So suppose (D2) holds, so that there is a nonzero Casimir $C \in S(\mathfrak{b})$. Transitivity implies that $\rho(C) = c$ for some constant $c \neq 0$. By the definition of a Casimir and the above remarks $C \in K$. Then $C - c \notin K$, for otherwise $c \in K$, which is at odds with the properness of K. But this contradicts the fact that $C - c \in \ker \rho \subset K$, and the theorem is proven.

Thus Conjecture 1 is consistent with the results listed in Table 2. Furthermore, by generalizing the construction of the polynomial quantization on $T^*\mathbf{R}_+$, one sees that the hypotheses of Conjecture 1 are certainly *necessary*.

Theorem 10 *Suppose that the polynomial algebra $P(\mathfrak{b})$ satisfies neither condition* (D1) *nor* (D2)*. Then any nontrivial quantization of $\mathfrak{b}$ extends to a quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Lastly, I observe that the finite-dimensionality assumption on $\mathfrak{b}$ in Conjecture 1 is necessary as well: The symmetric algebra $S(\mathfrak{t})$ on T^2 certainly contains Casimirs, but the conjecture is violated.

Establishing the validity of Conjecture 1 is the main goal of this proposal. Theorems 9 and 10 strongly suggest that the conjecture is correct (and, if not, then surely not too far off the mark). Still, completing its proof—that is, when M is noncompact and the quantizations are infinite-dimensional—seems to be a difficult problem. In particular, the proof of Theorem 9 (as well as that of Theorem 3) relies on Lemma 1, which in turn is predicated on the finite-codimensionality of $\ker \mathcal{Q}$. As there is no apparent reason why this should be so under the stated circumstances, different techniques will be required to handle this case.

For instance, here is an "algebraic no-go theorem" which does not require that the codimension of $\ker \mathcal{Q}$ be finite [GGra1]:

Theorem 11 *Let $\mathcal{P}$ be a unital Poisson subalgebra of $C^\infty(M, \mathbf{C})$. If as a Lie algebra $\mathcal{P}$ is not commutative, it cannot be realized as an associative algebra with the commutator bracket.*

To apply this result to polynomial quantizations, suppose that $\mathcal{Q} : P(\mathfrak{b}) \to \mathrm{Op}(D)$ were a quantization of $(P(\mathfrak{b}), \mathfrak{b})$ on some invariant dense domain D in a Hilbert space. By requiring $\mathcal{Q}$ to be complex linear, we may regard it as a quantization of $\mathcal{P} = P(\mathfrak{b})_{\mathbf{C}}$. Define $\mathcal{A} \subset \mathrm{Op}(\mathcal{D})$ to be the associative algebra generated over $\mathbf{C}$ by $\{\mathcal{Q}(f) \mid f \in \mathfrak{b}\}$ together with I (if $1 \notin \mathfrak{b}$). If it can be shown that any such $\mathcal{Q}$ must be a Lie algebra isomorphism of $P(\mathfrak{b})$ onto $\mathcal{A}$, then Theorem 11 will yield a contradiction. One can use this result to give an alternate proof of Groenewold's theorem; in fact, it is the key ingredient in the proof of Theorem 6 [GGra1].

It may be necessary to work through a few more examples before one is able to gain sufficient insight into this problem. One example worth studying are the various coadjoint orbits for the symplectic algebra $\mathrm{sp}(2n, \mathbf{R})$. Another would be arbitrary cotangent bundles (although then $\mathfrak{b}$ will typically be infinite-dimensional). As well, it would be useful to consider basic algebras of a more general type than the ones we have encountered thus far (which were all either solvable or semisimple). I have also restricted consideration to polynomial algebras to a large extent, but there are other subalgebras $\mathcal{O}$ of $C^\infty(M)$ which are of interest (e.g., on $\mathbf{R}^{2n}$, those functions which are constant outside some compact set [Ch3]).

Of our six examples, the torus is clearly much different than the others. Because the basic algebra $\mathfrak{t}$ is infinite-dimensional, the irreducibility requirement (Q4) loses much of its force—so much so that it precludes the existence of an obstruction. So it seems equally reasonable to propose

Conjecture 2 *Let M be a symplectic manifold and $\mathfrak{b}$ a basic algebra with $P^1(\mathfrak{b})$ dense in $C^\infty(M)$. Then there exists a nontrivial quantization of $(C^\infty(M), \mathfrak{b})$.*

A necessary condition for $\mathcal{Q}$ to be a full quantization of $(C^\infty(M), \mathfrak{b})$ is that $\mathcal{Q}$ represent $C^\infty(M)$ itself irreducibly. It turns out [Ch2, Tu] that this is so for all Kostant-Souriau prequantizations[5]; thus it is natural to consider the case when M is prequantizable in this sense. In fact, in this context [Tu] gives even more:

Proposition 12 *Let M be an integral symplectic manifold, L a Kostant-Souriau prequantization line bundle over M and $\mathcal{Q}_L$ the corresponding prequantization map. Let $\mathfrak{b}$ be a basic algebra with $P^1(\mathfrak{b})$ dense in $C^\infty(M)$. Then $\mathcal{Q}_L$ represents $\mathfrak{b}$ irreducibly.*

Let $D_c = \Gamma(L)_c$, the compactly supported sections of L. By construction $\mathcal{Q}_L : C^\infty(M) \to \mathrm{Op}(D_c)$ satisfies (Q1)–(Q3) and (Q6). This proposition guarantees that $\mathcal{Q}_L$ satisfies (Q4) as well. Thus to obtain a full quantization it remains to verify (Q5) on some appropriately extended domain D; unfortunately, it does not seem possible to do this except in specific instances. Noting that the quantization of T^2 given in §C.I is actually a Kostant-Souriau prequantization with Chern number $N = 1$ [Go4], a first test would be to understand what happens for the Kostant-Souriau prequantizations of $\big(C^\infty(T^2), \mathfrak{t}\big)$ with $N \neq 1$. Still, Proposition 12 does provide a certain amount of support for Conjecture 2.

The "gray area" between Conjectures 1 and 2 consists of symplectic manifolds with basic algebras $\mathfrak{b}$ for which $P^1(\mathfrak{b})$ is infinite-dimensional, yet not dense in $C^\infty(M)$. Perhaps the infinite-dimensionality of $\mathfrak{b}$ alone is enough to guarantee the existence of a full quantization? In any event, it might be fruitful to search for more examples of full quantizations.

Setting aside the question of the existence of obstructions, I now suppose that there is an obstruction to, say, a polynomial quantization, so that it is impossible to consistently quantize all of $P(\mathfrak{b})$. The question then is: *What are the largest Lie subalgebras $\mathcal{O} \subset P(\mathfrak{b})$ containing the given basic algebra $\mathfrak{b}$ such that $(\mathcal{O}, \mathfrak{b})$ can be quantized?* Modulo technical issues, given a representation $\mathcal{Q}$ of $\mathfrak{b}$ on a Hilbert space $\mathcal{H}$, one ought to be able to induce a representation of its Lie normalizer $\mathfrak{n}(\mathfrak{b})$ in $P(\mathfrak{b})$ on $\mathcal{H}$. (Indeed, the structure $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$ brings to mind an infinitesimal version of a Mackey system of imprimitivity [Ma].) Thus it seems reasonable to assert:

Conjecture 3 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra. Then every quantization of $\mathfrak{b}$ can be extended to a quantization of $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$.*

[5] However, there are other prequantizations which do not represent $C^\infty(M)$ irreducibly; for instance, the prequantization of Avez [Av2, Ch3].

This is in exact agreement with the examples. In particular, for $\mathbf{R}^{2n}$ one has $\mathfrak{n}(P^1) = P^2$, and for S^2 one computes $\mathfrak{n}(P_1) = P^1$. Moreover, in both cases we have shown that these normalizers are in fact maximal quantizable Lie subalgebras of polynomials [Go6, GGH]. It is therefore tempting to conjecture that no nontrivial quantization of $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$ can be extended beyond $\mathfrak{n}(\mathfrak{b})$ [GGT], which is indeed the case for both $\mathbf{R}^{2n}$ and S^2. If true, this would also point where to look for a Groenewold-Van Hove contradiction, viz. just outside the normalizer. Alas, this is false: For the cylinder $\mathfrak{n}(P_1) = P^1$. But in [GGru1] we showed that any representation $\mathcal{Q}(P_1)$ can be extended, in infinitely many ways, to quantizations of (L^1, P_1), where L^1 is the Lie subalgebra of polynomials which are affine in the (angular) momentum. It is not clear how one could "discover" this subalgebra given just the basic algebra $\mathfrak{e}(2)$ (but see below). An outstanding problem is therefore to determine the largest Lie subalgebras of quantizable observables. Unfortunately, this is extremely difficult in general. Just classifying maximal Lie subalgebras of polynomials is daunting; for instance, this problem has not been solved even for $\mathbf{R}^{2n}$ when $n > 1$ [Go6].

This is reminiscent of the situation in geometric quantization with respect to polarizations. Suppose that $\mathcal{A}$ is a polarization of $C^\infty(M, \mathbf{C})$. Then one knows that one can consistently quantize those observables which preserve $\mathcal{A}$, i.e., which belong to the real part of $\mathfrak{n}(\mathcal{A})$ [Bl1, Wo]. In this way one obtains a "lower bound" on the set of quantizable functions for a given polarization. If one takes the antiholomorphic polarization on S^2, then it turns out that the set of *a priori* quantizable functions obtained in this manner is precisely P^1. But it may happen that the real part of $\mathfrak{n}(\mathcal{A})$ is too small, as for $\mathbf{R}^{2n}$ with the antiholomorphic polarization. In this case the real part of $\mathfrak{n}(\mathcal{A})$ is only a proper subalgebra of P^2, and in particular is not maximal. This reflects the fact that the metaplectic representation cannot be derived by polarizing a prequantization. Furthermore, in the case of the torus, introducing a polarization will drastically cut down the set of *a priori* quantizable functions, which is at odds with the existence of a full quantization of this space. So geometric quantization is not a reliable guide insofar as computing maximal quantizable Lie subalgebras of observables. On the other hand, the subalgebra L^1 is just the normalizer of the vertical polarization $\{h(\theta)\}$ on T^*S^1, so this subalgebra finds a natural interpretation in the context of polarizations. Clearly, there must be some connection between polarizations and basic algebras that awaits elucidation.

Assuming that one can in fact determine the largest Lie algebras of quantizable observables, the next problem is to *classify all their possible quantizations.* Again, this can only be done on a case-by-case basis at present. However, some interesting phenomena already have come to light in those examples which have been analyzed in detail. For instance, if C denotes the Lie subalgebra of polynomials on $\mathbf{R}^{2n}$ which are at most affine in the momenta, then it turns out [Go6] that (C, P^1) can be consistently quantized, and moreover that *any* quantization thereof is unitarily equivalent to $\mathcal{Q}_\eta$ defined by

$$\mathcal{Q}_\eta\left(\sum_{i=1}^n f^i(q)p_i + g(q)\right) = -i\hbar \sum_{i=1}^n \left(f^i(q)\frac{\partial}{\partial q^i} + \left[\frac{1}{2} + i\eta\right]\frac{\partial f^i(q)}{\partial q^i}\right) + g(q), \tag{4}$$

where f^i, g are polynomials and $\eta \in \mathbf{R}$. When $\eta = 0$, this is the familiar "position" or "coordinate representation." But for $\eta \neq 0$ the $\mathcal{Q}_\eta$ are genuinely new quantizations, and it evidently is of interest to see if they have any observable consequences or physical applications. The significance of the parameter η is explored in [ADT, Hen]. There it is shown that while the quantizations $\mathcal{Q}_\eta$ and $\mathcal{Q}_{\eta'}$ are unitarily inequivalent if $\eta \neq \eta'$, they are related by a nonlinear norm-preserving isomorphism. Similarly the physical significance of the full quantization (1) of $C^\infty(T^2)$ remains unclear; it is interesting that this quantization appears (in a completely different context) in solid state physics, where it is known as the "kq-representation" [Za].

Apropos the remarks above, it would clearly be worthwhile, presuming that it is somehow possible to predict the maximal set(s) of quantizable observables *a priori*, to see whether one can use this knowledge to refine geometric quantization theory, or to *develop a new quantization procedure, which is adapted to the Groenewold-Van Hove obstruction in that it will automatically be able to quantize this maximal set of observables.* In this regard, I note that many of the quantizations we have computed (such as the $\mathcal{Q}_\eta$ of (C, P^1) given by (4) with $\eta \neq 0$, or the metaplectic quantization of (P^2, P_1), both on $\mathbf{R}^{2n}$) cannot be derived using geometric quantization theory. Again, a first step here would be to recast the Groenewold-Van Hove results in terms of polarizations. Such a quantization procedure would be of obvious value when one is confronted with complicated classical systems, such as homogeneous cosmological models [GI], whose quantum analogues are at best poorly understood.

Here I have focused on the quantization of symplectic manifolds. It is natural to wonder to what extent these results will carry over to Poisson manifolds, or even to abstract Poisson algebras.

My approach is designed so as to obtain results which are independent of the particular quantization scheme employed, as long as it is Hilbert-space based. Therefore it is interesting that some of the go and no-go results described here have direct analogues in deformation quantization theory, since this theory was developed, at least in part, to avoid the use of Hilbert spaces altogether [BFFLS]. So for example [Ri1], the no-go result for S^2 is mirrored by the fact that there are no strict SU(2)-invariant deformation quantizations of $C^\infty(S^2)$, while the go theorem for T^2 has as a counterpart the result that there do exist strict deformation quantizations of the torus. Since every symplectic manifold admits a (formal) deformation quantization [Ko], it is sometimes asserted that the existence of Groenewold-Van Hove obstructions necessitates a weakening of the Poisson bracket $\rightarrow$ commutator rule (by insisting that it hold only to order $\hbar$). This is far from clear, however, for two reasons. First, this very fact indicates that in general a deformation quantization is not a "quantization" in any true physical sense, since certainly there are symplectic manifolds which cannot be recovered in the limit as $\hbar \rightarrow 0$ from any quantum system. What is probably required in this context is not merely a formal deformation quantization, but rather an appropriately invariant strict deformation quantization [Ri1, Ri3]; see also [Fr]. Second, as the observations above indicate, weakening the Poisson bracket $\rightarrow$ commutator rule may not suffice to remove the obstructions. There are undoubtedly important things to be learned by getting to the heart of this analogy.

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Two colleagues will be collaborating with me on my proposed research project: Prof. Janusz Grabowski (Mathematics Institute, University of Warsaw, Warsaw, Poland), an expert in the field of Poisson algebras, and Prof. Hendrik Grundling (Pure Mathematics, University of New South Wales, Sydney, Australia), a mathematical physicist specializing in quantum field theory and functional analysis. Both have collaborated with me in the past on my NSF-sponsored research.

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E. Biographical Sketch

a. Professional Preparation

B.S. Summa Cum Laude with Distinction in Physics (Duke University, 1973)
M.S. in Physics (University of Maryland, 1975)
Ph.D. in Physics (University of Maryland, 1979)
Research Associate in Applied Mathematics (University of Calgary, 1979–1981)

b. Appointments

1. Professor, 1992-present, Department of Mathematics, University of Hawai'i at Manoa, 2565 The Mall, Honolulu, HI 96822 (Email: gotay@math.hawaii.edu, Home Page: www.math.hawaii.edu/~gotay)
2. Ford Foundation Fellow, 1988-1989, Mathematical Sciences Research Institute, Berkeley, CA 94720
3. Associate/Assistant Professor, 1987-1992/1984-1987, Mathematics Department, United States Naval Academy, Annapolis, MD 21402
4. Assistant Professor, 1981-1984, Department of Mathematics, University of Calgary, Calgary, AB Canada T2N 1N4

Honors and Awards:

Visiting Researcher, Bulgarian Academy of Sciences, July 1999; National Science Foundation International Supplement, 1997; National Science Foundation Grant, 1996-2000; Visiting Professor, University of New South Wales, Summer 1993; Chairman of the Special Session on Symplectic Geometry at the International Joint Mathematics Meetings (Vancouver, August, 1993); National Science Foundation Grant, 1992-1995; Chairman of the Organizing Committee of the AMS/IMS/SIAM Joint Summer Research Conference on Mathematical Aspects of Classical Field Theory (Seattle, July, 1991); Elected to the Editorial Board of the journal *Differential Geometry and Its Applications*, 1991; Visiting Professor, Université d'Aix-Marseille, June 1990; Mathematics Department Researcher of the Year Awards (U.S. Naval Academy), 1992, 1991, 1990; Member, Mathematical Sciences Research Institute, 1988-1989; National Science Foundation Grant, 1988-1992; U.S. Naval Academy Faculty Performance Awards, 1990, 1988; Ford Foundation Postdoctoral Fellowship, 1988-1989; ONR/Naval Academy Research Council Grants, 1992, 1987, 1986, 1985, 1984; Sigma Xi, 1985; Honorable Mention for the essay, *Time and Singularity*, Gravity Research Foundation Awards for Essays on Gravitation Competition, 1983; University of Calgary/NSERC grants, 1983, 1980; Forth Prize for the essay, *Quantum Cosmology and Geometric Quantization*, Gravity Research Foundation Awards for Essays on Gravitation Competition, 1980; Ford Foundation Postdoctoral Fellowship (declined), 1980; NASA Traineeships in Space Science, NASA Headquarters, Washington, DC, 1979 (declined), 1978, 1977, 1976.

c. Publications

(i) 5 Publications Related to Proposed Research:

1. M.J. Gotay & J. Grabowski, *On Quantizing Nilpotent and Solvable Basic Algebras*, submitted to Bull. Can. Math. Soc. (1999) math-ph/9902012.
2. M.J. Gotay, J. Grabowski, & H.B. Grundling, *An obstruction to quantizing compact symplectic manifolds*, Proc. Amer. Math. Soc. **128**, 237–243 (2000).

3. M.J. Gotay, *Obstructions To Quantization,* to appear in the *Juan Simo Memorial Volume*, J.E. Marsden & S. Wiggens, Eds. (Springer, 1999) math-ph/9809011.
4. M.J. Gotay, H. Grundling & C.A. Hurst, *A Groenewold-Van Hove Theorem for* S^2, Trans. Amer. Math. Soc. **348**, 1579–1597 (1996).
5. M.J. Gotay, *On a Full Quantization of the Torus*, in *Quantization, Coherent States and Complex Structures*, J.-P. Antoine et al., Eds. 55–62 (Plenum, 1995).

(ii) 5 Other Significant Publications:

1. M.J. Gotay & J.E. Marsden, *Stress-Energy-Momentum Tensors and the Belinfante-Rosenfeld Formula*, in: *Mathematical Aspects of Classical Field Theory,* M.J. Gotay, J.E. Marsden & V.E. Moncrief, Eds., Contemp. Math. **132**, 367-392 (1992).
2. J.M. Arms, M.J. Gotay & G. Jennings, *Geometric and Algebraic Reduction for Singular Momentum Mappings*, Adv. in Math. **79**, 43-103 (1990).
3. M.J. Gotay, *Constraints, Reduction and Quantization,* J. Math. Phys. **27**, 2051-2066 (1986).
4. M.J. Gotay, *On Coisotropic Imbeddings of Presymplectic Manifolds,* Proc. Amer. Math. Soc. **84**, 111-114 (1982).
5. M.J. Gotay, J.M. Nester, & G. Hinds, *Presymplectic manifolds and the Dirac-Bergmann theory of Constraints*, J. Math. Phys. **19**, 2388–2399 (1978).

e. Collaborators & Other Affiliations

(i) Collaborators

Jacques Demaret (Institut d'Astrophysique, Université de Liège) [deceased], Janusz Grabowski (University of Warsaw), Hendrik Grundling (University of New South Wales), C. Angas Hurst (University of Adelaide), James A. Isenberg (University of Oregon), Jerrold E. Marsden (Caltech), Ivailo Mladenov (Bulgarian Academy of Sciences), Gijs M. Tuynman (Université de Lille I).

(ii) Graduate and Postdoctoral Advisors

Dissertation Advisor: Robert H. Gowdy (Virginia Commonwealth University);
Postdoctoral Sponsor: Jędrzej Śniatycki (University of Calgary)

(iii) Thesis Advisor and Postgraduate-Scholar Sponsor

Graduate Students: (1) Bryon Kaneshige (University of Hawai'i)

f. Other Personnel

Two colleagues will be collaborating with me on my proposed research project: Prof. Janusz Grabowski (Mathematics Institute, University of Warsaw, Warsaw, Poland), an expert in the field of Poisson algebras, and Prof. Hendrik Grundling (Pure Mathematics, University of New South Wales, Sydney, Australia), a mathematical physicist specializing in quantum field theory and functional analysis. A graduate student, Bryon Kaneshige, also intends to do his dissertation in the proposed area of research.

Instructions to NSF Submitters: The functions below work on this spreadsheet only.
These functions may not work properly if you add any columns.
These functions will add the necessary labels required for each row.

Notes: If you delete a row created with the buttons, prior to using the buttons again you must remove the labels that were created with the r
Do not enter anything in the red cells unless you intend to override the formula values.
You should be entering the values you want Fastlane to receive in the yellow cells.
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If you want to create additional years based on the data in a first year, then do not press the "create new budget years" button until y
The "create new budget years" button will copy all data from budget year1 to subsequent years.
Comments fields are limited to 255 characters - please do not enter more than that.
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YEAR 1

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ORGANIZATION **Your Organization goes here**	PROPOSAL NO.	DURATION (MONTHS) Proposed	Granted
PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR	AWARD NO.	Funds Granted by NSF	

A. SENIOR PERSONNEL: PI/PD, Co-PI´S, Faculty and Other Senior Associates (List each separately with title, A.7. show number in brackets)	NSF Funded Person-months CAL	ACAD	SUMR	Funds Requested By Proposer	Funds Granted by NSF	COMMENTS
0. First Name / M / Last Name / Title						
1. FirstNamePI X LastNamePI Title	0.00	0.00	0.00	$0		
(1) TOTAL SENIOR PERSONNEL (1-6)				$0		
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)						
1. (0) POST DOCTORAL ASSOCIATES	0.00	0.00	0.00	$0		Post Doc:
2. (0) OTHER PROFESSIONALS (TECHNICIAN, PROGRAMMER, ETC.)	0.00	0.00	0.00	$0		Other Professionals:
3. (0) GRADUATE STUDENTS				$0		Grad Students:
4. (0) UNDERGRADUATE STUDENTS				$0		Undergrad:
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)				$0		Secretarial:
6. (0) OTHER				$0		Other Persons:
TOTAL SALARIES AND WAGES (A+B)				$0		Total Salaries:
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				$0		Fringe Benefits:
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$0		Total Salary and Fringe:
D. PERMANENT EQUIPMENT (LIST ITEM AND DOLLAR AMOUNT FOR EACH ITEM EXCEEDING $5,000)						
equipment item 1 $0						Equipment comments:
TOTAL EQUIPMENT				$0		
E. TRAVEL 1. DOMESTIC (INCL. CANADA, MEXICO AND U.S. POSSESSIONS)				$0		Domestic Travel:
2. FOREIGN				$0		Foreign Travel:
F. PARTICIPANT SUPPORT COSTS						
1. STIPENDS $0						Stipends:
2. TRAVEL $0						Travel:
3. SUBSISTENCE $0						Subsistence:
4. OTHER $0						Other:
(0) TOTAL NUMBER OF PARTICIPANTS				$0		
G. OTHER DIRECT COSTS						
1. MATERIALS AND SUPPLIES				$0		Materials:
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION				$0		Publications:
3. CONSULTANT SERVICES				$0		Consultant:
4. COMPUTERS SERVICES				$0		Computer:
5. SUBAWARDS				$0		Subawards:
6. OTHER				$0		Other:
TOTAL OTHER DIRECT COSTS				$0		Other Total:
H. TOTAL DIRECT COSTS (A THROUGH G)				$0		Direct Total:
I. INDIRECT COSTS (SPECIFY RATE AND BASE)						
Name of indirect cost item / Amount / Rate						
FirstIndirectCostItem $0 0% 0						Indirect Cost Items:
TOTAL INDIRECT COSTS				$0		
J. TOTAL DIRECT AND INDIRECT COSTS (H+I)				$0		
K. RESIDUAL FUNDS (IF FOR FURTHER SUPPORT OF CURRENT PROJECTS SEE GPG II.D.7.j.)				$0		
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				$0		
M. COST SHARING: PROPOSED LEVEL AGREED LEVEL IF DIFFERENT $				$0		

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INST. REP. TYPED NAME & SIGNATURE*	DATE	Date Checked	Date Of Rate of Sheet

Budget Explanation Page

A. <u>Senior Personnel Salary</u>

The budget covers two months summer salary for the Principal Investigator for each year of the project.

B. <u>Other Personnel Salary</u>

The budget includes half-time support during the academic year plus one month summer support for a graduate student for each year of the project.

C. <u>Fringe Benefits</u>

Fringe benefits are assessed at 3.50% for summer overload for senior personnel and graduate students alike. They are assessed at 25.00% for graduate students for support during the academic year.

E. <u>Travel</u>

Domestic: Will be used to attend a North American conference every year (e.g., A.M.S. or specialized meetings), and/or to visit colleagues at various institutions for the purpose of research collaboration, etc.

Foreign: In my field, the most important conferences are usually held in Europe. As well, many of the prominent researchers in my field, including my collaborators on the proposed project, are located overseas. Thus I have planned for one extended overseas trip each year.

[Remark: Airfare to and from Hawaiʻi is considerably more expensive than fares originating within the mainland U.S.]

G. <u>Other Direct Costs</u>

Materials and Supplies: This will cover long distance charges, book purchases, computer supplies and software, etc. In particular, I periodically need to either acquire and/or update mathematical software (e.g., *Mathematica* and *Textures*) which is essential for my research.

Consultant Services: This will be used to defray the living and travel expenses of my research collaborators (cf. the proposal, Proposal Section I. Special Information and Supplementary Information) when they come to with work me in Hawaiʻi. I anticipate one such visit per year. These funds will also be used to partially support other colleagues who visit me.

List of Suggested Reviewers

Colleagues whom I feel would be appropriate to referee this proposal include: ...

I have collaborated with ... in the past, but not on the proposed research. I will be happy to provide full addresses upon request.

List of Reviewers Not to Use:

....